Correspondences

Functions are single-valued. For a function $f : X \to Y$, every $x \in X$ is mapped to one *and only one* point $y \in Y$, the point $y = f(x)$. Thus, for example, when a consumer with a strictly quasiconcave utility function behaves according to the Utility Maximization Hypothesis, we can summarize his market behavior by his demand function: the bundle he chooses is a (single-valued) function of the price-list he faces.

But suppose the consumer's utility function is not strictly quasiconcave; for example, suppose it is $u(x_1, x_2) = ax_1 + x_2$. Whenever the price-list satisfies $p_1 = ap_2$, this consumer is indifferent among *all* the bundles on the indifference curve $\{\mathbf{x} \in \mathbb{R}^2_+ \mid ax_1 + x_2 = a\mathring{x}_1 + \mathring{x}_2\}$. He will of course choose a single bundle, but we can't say which one it will be. We can say only that it will be one of the many utility-maximizing bundles.

This situation, in which we need to analyze behavior that does not manifest itself in uniquely determined actions, is extremely common in economics and game theory. And of course, if individuals' behavior is not single-valued, then aggregate behavior won't be single-valued either. In the demand theory example above, the market demand function will not be single-valued. We need a new analytical tool, the *multivalued function* or *correspondence*.

Definition: A *correspondence* f from a set X to a set Y, denoted $f : X \twoheadrightarrow Y$, is a function from X to the set 2^Y of all subsets of Y. Correspondences are also called multivalued functions.

Remark: A correspondence $f : X \twoheadrightarrow Y$ is thus a set-valued function from X to Y — for every $x \in X$, $f(x)$ is a subset of Y.

Example 1: $f : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}$ is defined by $f(x) = \{\sqrt{x}, -\sqrt{x}\}$.

Example 2: $f : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} [0, \frac{1}{x}] & \text{if } x > 0 \\ \emptyset & \text{if } x = 0. \end{cases}$$

Example 3: $f : \mathbb{R}_+ \longrightarrow\!\!\!\!\to [0,1]$ is defined by

$$f(x) = \begin{cases} [x, \frac{1}{2}] & \text{if } 0 \leqq x \leqq \frac{1}{2} \\ \{\frac{1}{4}, \frac{1}{2}\} \cup [\frac{3}{4}, 1] & \text{if } x = \frac{1}{2} \\ \{\frac{1}{4}\} \cup [\frac{3}{4}, 1] & \text{if } x \geqq \frac{1}{2} \end{cases}$$

Definition: The *graph* of a correspondence $f : X \longrightarrow\!\!\!\!\to Y$, denoted $\mathrm{Gr}(f)$, is the set

$$\mathrm{Gr}(f) := \{(x, y) \in X \times Y \mid y \in f(x)\ \}.$$

Exercise: Draw $\mathrm{Gr}(f)$ for each of the three examples given above.

Two continuity properties are important for correspondences:

Definition: A correspondence $f : X \longrightarrow\!\!\!\!\to Y$ has a *closed graph* if $\mathrm{Gr}(f)$ is a closed subset of $X \times Y$.

Remark: A correspondence thus has a closed graph if and only if it satisfies the following condition: if $\{x_n\}$ and $\{y_n\}$ are sequences in X and Y such that $y_n \in f(x_n)$ for each n, and if $\{x_n\} \to x$ and $\{y_n\} \to y$, then $y \in f(x)$.

Definition: If Y is a compact set, a correspondence $f : X \longrightarrow\!\!\!\!\to Y$ is *continuous* if $\forall x \in X : f(x) \neq \emptyset$ and if f has a closed graph and also satisfies the following condition:

(LHC) for every $(x, y) \in \mathrm{Gr}(f)$ and every sequence $\{x_n\} \to x$ there is a sequence $\{y_n\}$ that satisfies both

$$(1) \quad \forall n : y_n \in f(x_n) \qquad \text{and} \qquad (2) \quad \{y_n\} \to y.$$

Correspondences that satisfy the condition (LHC) are called *lower hemicontinuous.* Thus, a correspondence with a compact target space is continuous if and only if it is nonempty-valued, has a closed graph, and is lower hemicontinuous.

Example 1 above has a closed graph and satisfies (LHC) but its target space is not compact. Example 2 satisfies (LHC) but does not have a closed graph. Example 3 is continuous.

Example 4: $f : [0,1] \twoheadrightarrow [0,1]$ is defined by

$$f(x) = \begin{cases} [.3,.7] & \text{if } x \leqq \frac{1}{2} \\ \{\frac{1}{2}\} & \text{if } x > \frac{1}{2}. \end{cases}$$

The correspondence in Example 4 has a closed graph but is not continuous (the condition (LHC) is violated).

Example 5: $f : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \{\frac{1}{x}\} & \text{if } x > 0 \\ \{0\} & \text{if } x = 0. \end{cases}$$

This correspondence is singleton-valued (each of its images $f(x)$ is a singleton). If we regarded its images as points rather than sets (remove the curly brackets in the definition), f would be a regular (single-valued) function from $\mathbb{R}_+$ into $\mathbb{R}$, and clearly a discontinuous function. But as a correspondence it is nonempty-valued, has a closed graph, and satisfies (LHC); however, it is not a continuous correspondence — its target space is not compact.

Exercise: Draw $\mathrm{Gr}(f)$ for Examples 4 and 5, and verify the claims made above about Examples 1 to 5 having closed graphs or not and satisfying (LHC) or not.

Definition: A fixed point of a correspondence $f : X \twoheadrightarrow Y$ is an $\widehat{x} \in X$ for which $\widehat{x} \in f(\widehat{x})$.

The most commonly used generalization of Brouwer's Theorem is Kakutani's Fixed Point Theorem:

Kakutani's Theorem: Let $f : S \twoheadrightarrow S$ be a correspondence. If S is nonempty, compact, and convex, and if f is nonempty-valued, convex-valued, and has a closed graph, then f has a fixed point.

When we say that f is nonempty-valued and convex-valued, we mean that $f(s)$ is a nonempty convex set for every $s \in S$.

Example 6: $f : [0,1] \twoheadrightarrow [0,1]$ is defined by

$$f(x) = \begin{cases} [.6, .8] & \text{if } x < \frac{1}{2} \\ [.2, .8] & \text{if } x = \frac{1}{2} \\ [.2, .4] & \text{if } x > \frac{1}{2}. \end{cases}$$

The correspondence in Example 6 is convex-valued and has a closed graph, and it therefore has a fixed point. Its unique fixed point is $\widehat{x} = \frac{1}{2}$.

Example 7: $f : [0,1] \twoheadrightarrow [0,1]$ is defined by

$$f(x) = \begin{cases} \{.7\} & \text{if } x < \frac{1}{2} \\ [.2, .4] & \text{if } x \geqq \frac{1}{2}. \end{cases}$$

The correspondence in Example 7 has no fixed point. The correspondence is convex-valued but does not have a closed graph: it's discontinuous at $x = \frac{1}{2}$.

Example 8: $f : [0,1] \twoheadrightarrow [0,1]$ is defined by

$$f(x) = \begin{cases} [.6, .8] & \text{if } x < \frac{1}{2} \\ [.2, .4] \cup [.6, .8] & \text{if } x = \frac{1}{2} \\ [.2, .4] & \text{if } x > \frac{1}{2}. \end{cases}$$

The correspondence in Example 8 has no fixed point. The correspondence has a closed graph but is not convex-valued: it's convex-valued at every x except $x = \frac{1}{2}$.

Economic models of individual behavior generally assume optimizing behavior. We typically use the classical Weierstrass Theorem ("a continuous function on a compact set attains a maximum") to infer that there is actually an optimizing action available for the individual to choose. Is the individual's behavior continuous? That is, does the individual's chosen action respond continuously to changes in his environment? When we have a specific functional form for the objective function (*e.g.*, a Cobb-Douglas utility function), we can often obtain a closed-form expression for the behavioral function and determine directly whether it's continuous. Even when we can't obtain a closed-form behavioral function, if the objective function is differentiable we can generally apply the Implicit Function Theorem to establish continuity (and even differentiability) of the behavioral function.

The preceding paragraph implicitly assumed that the individual's behavior is described by a *single-valued* function — that the optimizing action is always unique. We now have correspondences at our disposal, so we can now deal as well with situations in which the optimizing action is not unique — in which the behavioral function is actually a correspondence. The Maximum Theorem is used pervasively in economics and game theory to infer that behavioral correspondences are continuous (in the sense that they have closed graphs) when the underlying objective function and environment satisfy certain conditions.

In applications of the Maximum Theorem, the set X in the statement of the theorem below is typically the action space; Y is the set of possible environments (the parameter space); the function u is the objective function; the correspondence φ describes how the set of available actions depends upon the environment; and μ is the behavioral correspondence, describing how the individual's actions depend upon the environment he faces. In demand theory, for example, X would be the consumption set (or a compact subset of it); Y would be the set of possible price-lists (and possibly wealth/income levels); u would be the consumer's utility function; φ would be the correspondence that determines the budget set from the market prices; and μ would be the consumer's demand correspondence.

The Maximum Theorem: Let X be a compact subset of $\mathbb{R}^l$; let Y be a subset of $\mathbb{R}^l$; let $u : X \times Y \to \mathbb{R}$ be a continuous function; and let $\varphi : Y \twoheadrightarrow X$ be a continuous correspondence. Then the correspondence $\mu : Y \twoheadrightarrow X$ defined by

$$\mu(y) = \{\ x \in \varphi(y) \mid x \text{ maximizes } u \text{ on } \varphi(y)\ \}$$

has a closed graph.

Note that the domain of the objective function u includes not only actions (consumption bundles, in the demand application) but parameters as well (price-lists, in the demand application). This seems odd at first glance. There are three observations to be made about this:

(1) In the demand theory application we typically assume that u is constant with respect to the parameters $y \in Y$ — the prices. So for this application a Maximum Theorem in which u depends only upon consumption bundles in X would be just fine.

(2) Including the prices as arguments of u does, however, allow us to analyze consumers whose utility depends upon prices as well as consumption levels.

(3) The theory of the firm is an example of an application where we need to have the objective function u depend upon the parameters. In this application, the elements of X are the firm's feasible production plans, *i.e.*, input-output combinations. The elements of Y are again price-lists. The objective function u is the firm's profit function — and note that profit does depend upon both the firm's choice of production plan (in X) and the market prices (in Y). The correspondence φ describes how the available plans depend upon prices — typically, this is assumed to be a constant correspondence: the production plans available to the firm depend upon its technological capabilities but not upon prices. And of course the correspondence μ describes how the firm's profit-maximizing choices of input and output levels depend upon the market prices — the firm's supply correspondence for outputs and its demand correspondence for inputs.

Some useful facts about correspondences are provided here. Their proofs are straightforward and are good exercises for understanding correspondences.

Theorem: If Y is compact and the correspondences $f : X \twoheadrightarrow Y$ and $g : X \twoheadrightarrow Y$ both have closed graphs, then the sum $f + g$ also has a closed graph, where $f + g$ is the correspondence defined by

$$(f+g)(x) := \{\, y_1 + y_2 \in Y \mid y_1 \in f(x) \quad \text{and} \quad y_2 \in g(x) \,\}.$$

Theorem: If Y and Z are compact and the correspondences $f : X \twoheadrightarrow Y$ and $g : Y \twoheadrightarrow Z$ both have closed graphs, then the composition $f \circ g$ also has a closed graph, where $f \circ g$ is the correspondence defined by

$$(f \circ g)(x) := g(f(x)) := \{\, z \in Z \mid \exists y \in Y : y \in f(x) \;\&\; z \in g(y) \,\} = \bigcup \{\, g(y) \mid y \in f(x) \,\}.$$